

Large deviations and portfolio optimization

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Abstract : Risk control and optimal diversification constitute a major focus in the finance and insurance industries as well as, more or less consciously, in our everyday life. We present a discussion of the characterization of risks and of the optimization of portfolios that starts from a simple illustrative model and ends by a general functional integral formulation. A major theme is that risk, usually thought one-dimensional in the conventional mean-variance approach, has to be addressed by the full distribution of losses. Furthermore, the time-horizon of the investment is shown to play a major role. We show the importance of accounting for large fluctuations and use the theory of Cramér for large deviations in this context. We first treat a simple model with a single risky asset that exemplifies the distinction between the average return and the typical return, the role of large deviations in multiplicative processes, and the different optimal strategies for the investors depending on their size. We then analyze the case of assets whose price variations are distributed according to exponential laws, a situation that is found to describe reasonably well daily price variations. Several portfolio optimization strategies are presented that aim at controlling large risks. We end by extending the standard mean-variance portfolio optimization theory, first within the quasi-Gaussian approximation and then using a general formulation for non-Gaussian correlated assets in terms of the formalism of functional integrals developed in the field theory of critical phenomena.

1 Introduction

Finance and insurance are all about risk, and risk is usually thought to be all encapsulated by variances and covariances. In practice however, variances and covariances are quite unstable with the existence of intermittent bursts of high volatilities. More generally, risk is embedded in the distribution of asset returns and not solely in variance. In finance and in fact in many of our actions, we have to ponder the choice between risky alternatives with different rewards. How to make the best choice(s)?

“Don’t put all your eggs in one basket” is a familiar adage. The idea is then to try to minimize somehow the risk by diversification, a procedure that is exemplified by the problem of portfolio selection and optimization in finance: how to choose a basket of assets which maximizes your profit and minimizes your risk?

While these considerations seem quite straightforward and reflect the common sense, putting them in practice provides some surprise. Indeed, the very problem of the quantification of the notion of risk does not find a simple general unambiguous answer. Even if the stochastic process describing the set of potential profits and losses is stationary and can thus be described by a well-defined probability distribution, the problem of ordering distributions does not have a unique answer. This means that, in general, there does not exist a unique universal measure of risk, a single number that can be attributed to a risky situation. This is in contrast to the much better known problem of ordering numbers. As soon as one has to deal with pairs of numbers, vectors, matrices, and even more so distributions, the problem of ordering is much harder and in many cases has not a unique answer. There is a vast literature that addresses this question dealing with utility functions [1, 2], stochastic dominance and so on [3].

In section 2, we first present a simple model of a portfolio made of one risky and one riskless asset, which allows us to introduce the key ideas, namely the importance of the time-horizon of the investment, the distinction between the average return and the typical return, the role of large deviations in multiplicative processes, and the different optimal strategies for the investors depending on their size. In section 3, we analyze the case of assets whose price variations are distributed according to exponential laws, a situation that is found to describe reasonably well daily

price variations. This parametric case is used to illustrate the portfolio optimization strategies that aim to control large risks. In section 4, we summarize first the standard mean-variance portfolio optimization theory. We then show how to extend it within the quasi-Gaussian approximation. Finally, we provide a general formulation for non-Gaussian correlated assets in terms of the formalism of functional integrals developed in the field theory of critical phenomena.

2 An exactly soluble case: one risky and one riskless assets

Let us assume that the financial world is only made of a riskless asset and a risky asset. Without loss of generality, the return of the riskless asset can be taken zero by a suitable change of frame. Time is discrete in units of time τ taken to be unity. The return over a unit time period τ of the risky asset is a random variable $-\infty < r < +\infty$ without correlations, taken from a distribution $P(r)$. An investor has a wealth equal to unity at $t = 0$. Should he invest a fraction f in the risky asset? What is the optimal fraction f ?

We start by treating this simple case because it is particularly useful to introduce the key concepts that will be useful for the general problem of the diversification and optimization of portfolios. The standard mean-variance approach comes about very naturally and this models allows us to discuss its limits and generalization, in particular to take into account the existence of large deviations.

2.1 General case

Here, we assume that the problem is “separable”, i.e. f is a constant independently of the wealth. A more general case consists in evolving f as a function of the wealth of the investor, to account for the phenomenon that the degree of risk aversion is a function of wealth. This is usually addressed using the formalism of utility functions, which is mathematical device that quantifies the degree of satisfaction of an investor. Within this approach, the optimal fraction becomes a function $f(S(t), t)$ of time and of the wealth $S(t)$.

The choice $f = 0$ corresponds to an investor who is absolutely opposed to taking any risk and keeps his entire wealth in cash (or invested at the riskless return). Other investors will recognize that taking some risks by investing in the risky asset might provide gain opportunities above the riskless return. But this has a cost, the risk, and the question is how to choose the optimal degree of risk/return.

Let us evaluate the value $S(t)$ of the portfolio comprising a fraction f of the risky asset after t time steps. If we note r_1, r_2, \dots, r_t the specific returns observed during these t time steps, the wealth becomes

$$S(t) = \prod_{j=1}^t \left(1 - f + f e^{r_j}\right) = \prod_{j=1}^t \left(1 + (e^{r_j} - 1)f\right). \quad (1)$$

This expression contains all the information, as long as we can enumerate all possible scenarios r_1, r_2, \dots, r_t . In the absence of correlation between the returns r_j , all the information is in the distribution $P(S(t))$ of the wealth. The problem thus amounts to determine $P(S(t))$ knowing the distribution $P(r)$ of the one-step-returns of the risky asset.

Consider first the situation where the returns r_j are small ($r_j \ll 1$ for all j), $e^{r_j} - 1 \simeq r_j$ and $1 + r_j f \simeq e^{r_j f}$. The expression (1) becomes

$$S(t) \simeq \prod_{j=1}^t e^{r_j f} = \exp\left(f \sum_{j=1}^t r_j\right). \quad (2)$$

$\frac{1}{f} \log S(t)$ is the sum of t random variables and the central limit theorem gives

$$P(S(t)) \simeq \frac{1}{S_t \sqrt{2\pi t \sigma^2}} \exp\left(-\frac{(\log S_t - m f t)^2}{2 t f^2 \sigma^2}\right), \quad (3)$$

where $m \equiv \langle r_j \rangle$ and $\sigma^2 \equiv \langle r_j^2 \rangle - m^2$. The mean return of this portfolio is $r(f) = m f$ and its variance per unit time is $v(f) = \sigma^2 f^2$.

In this Gaussian limit, the risk is completely captured by the single number $v(f)$. For a given return $r = m f$, we get

$$r = \frac{m}{\sigma} \sqrt{v}, \quad (4)$$

by eliminating f . This relationship (4) sums up the standard result of the celebrated Markovitz portfolio theory [4, 5], namely the more risk one takes (quantified by \sqrt{v} ,

the larger the expected return r . There is no best optimal strategy f but rather a continuum of possibilities offered to the investor and whose choice depends on his risk aversion.

In the expression (3), $S(t)$ behaves typically as $S(t) = e^{rt+\eta\sqrt{vt}}$, where η is a gaussian random variable with zero mean and unit variance. $S(t)$ becomes larger than one with almost certainty for times larger than the characteristic time

$$t^* = \frac{2v}{r^2}, \quad (5)$$

at which the mean return rt becomes comparable with the amplitude $\sqrt{2vt}$ of the fluctuations. Thus, the decision to invest a fraction of his wealth in the risky asset must take into account the time interval during which he decides to immobilize his investment [6]. If this interval is too short, the results will be dominated by the fluctuations and the risk is large. The investment in the risky asset becomes profitable only for time intervals larger than t^* . The existence of this temporal factor in the investment strategy can be embodied in the so-called Sharpe ratio defined by the ratio of the expected return over the standard deviation

$$Sharpe_1 = \frac{r}{\sqrt{v}}. \quad (6)$$

$Sharpe_1$ can also be expressed as $Sharpe_1 = \sqrt{2}[t^*(f)]^{-1/2}$. The Sharpe ratio depends on the time scale. A good investment strives to decrease the time scale $t^*(f)$ beyond which the return is not anymore sensitive to fluctuations, hence to increase the Sharpe ratio.

2.2 An illustrative case

Before addressing the question of large risks, let us make concrete the previous results by specifying the distribution $P(r_j)$ of the risky asset. We consider a binomial case which has the advantage of allowing for an exact treatment. The return of the risky asset is thus assumed to be either

$$e^{r_j} = \lambda > 0 \quad \text{with the probability } p, \quad (7)$$

or

$$e^{r_j} = 0 \quad \text{with the probability } 1 - p. \quad (8)$$

This particular case can be interpreted as a casino game and has first been studied by Kelly [7], in relation with the theory of information which allows one to derive naturally the mean-variance approach in the Gaussian world.

A unit currency, from which a fraction f is invested in the risky asset, becomes on average after one time step :

$$p[1 + (\lambda - 1)f] + (1 - p)[1 - f] = 1 + (p\lambda - 1)f . \quad (9)$$

It is clear that only the case $p\lambda - 1 > 0$ is interesting and lead to investment strategies with $f \neq 0$. The average return per time step is $\log[1 + (p\lambda - 1)f]$ while the average gain is $(p\lambda - 1)f$.

In this model, the scenario r_1, r_2, \dots, r_t reduces to the knowledge of the number j of times where a gain (7) was realized. The case (8) for a loss then occurred $t - j$ times. The expression (1) becomes

$$S_j(t) = \left(1 + (\lambda - 1)f\right)^j (1 - f)^{t-j} . \quad (10)$$

The probability that this wealth is realized is simply the binomial law

$$\mathcal{P}(j) = \binom{t}{j} p^j (1 - p)^{t-j} . \quad (11)$$

The problem of the optimizing the investment is completely captured by the two formulas (10) and (11). In the limit of large investment times, the Stirling formula leads to the following expression

$$\mathcal{P}(j) \simeq \frac{1}{\sqrt{2\pi tp(1-p)}} e^{-\frac{(j-pt)^2}{2tp(1-p)}} , \quad (12)$$

valid close to its maximum. This expression (12) is nothing but the translation of the central limit theorem applied to the Binomial law, where the sum is of the type $\sum_{i=1}^t x_i$ with $x_i = 1$ with probability p and $x_i = 0$ probability $1 - p$. Expressing j as a function of $S_j(t)$ in (10),

$$j = \frac{\log \frac{S_j(t)}{(1-f)^t}}{\log \frac{1+(\lambda-1)f}{1-f}} , \quad (13)$$

we obtain the (log-normal) distribution of portfolio wealths at time t

$$\mathcal{P}(S(t)) \simeq \frac{1}{\sqrt{2\pi tp(1-p)}} \exp\left(-\frac{(\log \frac{S(t)}{S_{pp}(t)})^2}{2tv(f)}\right) , \quad (14)$$

where

$$S_{pp}(t) = \left(1 + (\lambda - 1)f\right)^{pt} (1 - f)^{(1-p)t} \quad (15)$$

is the most probable value of $S(t)$, i.e. the wealth that maximize the probability. It simply corresponds to the most probable value of j equal to pt . The variance $v(f)$ is given by

$$v(f) = p(1 - p) \left(\log \frac{1 + (\lambda - 1)f}{1 - f} \right)^2. \quad (16)$$

In this Gaussian limit, the portfolio selection problem is completely embodied by the two variables: the average return $r(f)$ defined by $r(f) = \frac{\log S_{pp}(t)}{t}$, i.e. $S_{pp}(t) = e^{rt}$, giving

$$r(f) = p \log(1 + (\lambda - 1)f) + (1 - p) \log(1 - f), \quad (17)$$

and the variance $v(f)$. Eliminating f allows us to obtain the mean-variance representation $r(v)$ which has become standard in portfolio theory.

Figure 1 shows r and v as a function of f for $p = 1/2$ and $\lambda = 2.1$. These numerical values give an average gain (9) $p\lambda - 1$ of 5%. Figure 2 gives r as a function of v for the same values. For f small, $r \simeq (p\lambda - 1)f$ and $v \simeq p(1 - p)\lambda^2 f^2$, which retrieves the dependence $r \propto \sqrt{v}$ found in the preceeding section. Figure 3 shows as a function of f for $r \geq 0$ the behavior of $t^*(f)$ defined by (5) and of $Sharpe_1 = \sqrt{2}[t^*(f)]^{-1/2} = \frac{r}{\sqrt{v}}$ defined for a unit time step. It is profitable to invest in the risky asset only if one is ready to freeze the investment over a period larger than $t^*(f)$. This time dimension is often forgotten or dismissed in the more standard approach that does not distinguish between the most probable and the mean gain. If the investment period t is fixed, the best portfolios correspond to those with $t^*(f) < t$. This introduces the constraint $r > \sqrt{\frac{v}{t}}$.

We observe that, for $p\lambda > 1$, r exhibits a maximum

$$r_{max} = p \log \lambda p + (1 - p) \log \frac{\lambda(1 - p)}{\lambda - 1} \quad (18)$$

reached for an optimal fraction f^* invested in the risky asset which is equal to

$$f^* = \frac{p\lambda - 1}{\lambda - 1}. \quad (19)$$

For $p = 1/2$ and $\lambda = 2.1$, $f^* = 4.54\%$, $r_{max} = 0.113\%$ and $v(f^*) = 0.00227$. The volatility \sqrt{v} , which gives the order of magnitude of the fluctuations of r , is equal to

4.76%, i.e. $t^*(f^*) \simeq 3500\tau$ and $Sharpe_1(f^*) = 0.024$. For $p = 1/2$ and $\lambda = 2.5$, $t^*(f^*)$ becomes smaller than 200τ , corresponding to $Sharpe_1 = 0.07$, with an optimum return better than 2%. For $p = 0.9$ and $\lambda = 1.2$, $t^*(f^*)$ is also of the order of 200τ , corresponding to $Sharpe_1 = 0.07$, with a similar optimum return 2%. As $p\lambda$ becomes much larger than 1, f^* increases as it becomes more and more interesting to invest in the risky asset. For $p = 1/2$ and $\lambda = 5$, $r_{max} = 2.5\%$, $f^* \simeq 0.4$, $t^*(f^*) \simeq 20$ and $Sharpe_1 = 0.25$. For $\lambda \rightarrow \infty$ and in the regime of positive mean returns, $Sharpe_1$ becomes independent of f and of λ : $Sharpe_1 \rightarrow \sqrt{\frac{p}{1-p}}$. It becomes completely controlled by the stochastic part of the process.

The existence of a maximum (18) was not predicted by the linear analysis of the previous section. It stems, as we now analyze, from the existence of large deviations inherent in the multiplicative processes (1,10).

2.3 Large risks and large deviations

2.3.1 Typical versus average returns

It is a well-known fact that average and typical values can be drastically different in multiplicative processes. This difference is at the origin of the existence of a maximum (18).

Let us come back to the exact expressions (10) and (11). The mathematical expectation $\langle S(t) \rangle$ of the value $S(t)$ of the portfolio corresponds to taking the average over a large number of realizations, in other words over a large number of investors. This will be a useful quantity for a bank, say, as opposed to a single investor. It is given by

$$\langle S(t) \rangle = \sum_{j=0}^t \mathcal{P}(j) S_j(t). \quad (20)$$

This expression can be summed and yields

$$\langle S(t) \rangle = \left(p[1 + (\lambda - 1)f] + (1 - p)[1 - f] \right)^t = \left((p\lambda - 1)f + 1 \right)^t. \quad (21)$$

We thus simply retrieve the expression (9) taken to the power t , i.e.

$$(1 + \langle \text{gain over one time step} \rangle)^t. \quad (22)$$

$\langle S(t) \rangle$ is not the same as the most probable value $S_{pp}(t)$ given by (15). $\langle S(t) \rangle$ and $S_{pp}(t)$ have very different behaviors. Figure 4 gives the dependence of $\langle S(1) \rangle$ and $S_{pp}(1)$ as a function of f for the case $p = 1/2$ and $\lambda = 2.1$ studied previously. As can be seen from (21), $\langle S(1) \rangle$ is a linear function of f while $S_{pp}(1)$ exhibits a highly nonlinear dependence. Technically, the difference can be tracked back to the fact that $S_{pp}(t)$ is the exponential of the average of the logarithm of $S(t)$, i.e. the geometrical mean of $S(t)$:

$$S_{pp}(t) = e^{\langle \log S(t) \rangle}, \quad (23)$$

which can be checked by calculating $\langle \log S(t) \rangle = \sum_{j=0}^t \mathcal{P}(j) \log S_j(t)$. This is to be compared to

$$\langle S(t) \rangle = \langle e^{\log S(t)} \rangle, \quad (24)$$

which is the arithmetic average. For f small for which the return is close to zero, the two values become indistinguishable, as can be seen in figure 4. $f \ll 1$ implies small fluctuations of wealth and the two averages become identical in this limit.

In contrast for arbitrary f , $\langle \log S(t) \rangle$ and $\log \langle S(t) \rangle$ can be very different. Mathematically, the exchange between the mean and the exponentiation is not justified. Expanding the exponential in (24) yields

$$\langle S(t) \rangle = 1 + \langle \log S(t) \rangle + \frac{1}{2} \langle (\log S(t))^2 \rangle + \dots, \quad (25)$$

that can be compared with

$$S_{pp}(t) = e^{\langle \log S(t) \rangle} = 1 + \langle \log S(t) \rangle + \frac{1}{2} \langle \log S(t) \rangle^2 + \dots. \quad (26)$$

The difference between the two quantities occurs at the second order in the expansion. More generally, for any positive random variable, $\langle x^k \rangle > \langle x \rangle^k$ and thus $\langle S(t) \rangle > S_{pp}(t)$. As an illustration, figure 4 shows that $\langle S(1) \rangle \simeq 1.002$ while $S_{pp}(1) \simeq 1.001$ at the maximum of the mean return. After 1000 time steps, this leads to $\langle S(1000) \rangle \simeq 7.4$ while $S_{pp}(1000) \simeq 2.7$. The difference explodes exponentially at large times. Notice that, for $f > 0.09$, r becomes negative and thus $S_{pp}(t)$ decreases exponentially while $\langle S(t) \rangle$ still increases exponentially: a typical investor will loose almost surely while very rare investors will exhibit shameless gains of such a magnitude as to make the average economy still growing!

The origin of these differences can be found in the large fluctuations of multiplicative process, exhibiting rare fluctuations but of sufficient magnitude to control the behavior of the mean gain. For instance, a sequence of positive returns such that there are $j = t$ successes occurs with an exponentially small probability. However, its contribution to the mean gain has an exponentially large magnitude. As a consequence, this type of scenarios brings in an important contribution to $\langle S(t) \rangle$ and a dominant contribution of higher moments.

The difference between (23) and (24) illustrates dramatically the importance of diversification. The result (24) quantified the results obtained by an investor A who could divide his initial wealth in N independent assets, each asset corresponding to the couple (f invested in the risky game, $1 - f$ kept at the riskless return), each playing a given game independently of the others. This diversification is gratifying as his total wealth is controlled by the mean gain, this becoming true in the limit of large N . This is very different from the result (23) obtained by an investor B who has only a single asset and is thus completely controlled by a single scenario. In practice, for the wealth to be described by (24), the number of assets N must be exponentially large in t [8]: $N \propto e^{ct}$, where $c > 0$ is an increasing function of f . At short times, it is not very useful to diversify, but as time passes by, the diversification becomes essential. This introduces an additional level of complexity: for practical implementation, N can not be arbitrarily large; one can thus imagine investment strategies which modify f such that the number $N(t)$ necessary to track the mean gain remains bounded.

These effects are well-known in the physics of disordered media and are at the origin of a wealth of effects, culminating in the non-ergodic behavior of spin-glasses. The consequence of large fluctuations in multiplicative processes in the context of portfolio selection has also been stressed recently [9].

2.3.2 Large deviations

Large risks are characterized by the existence of deviations from the behavior described by the central limit theorem.

Indeed, the previous considerations have relied on the fact that the fluctuations of

the return could be adequately quantified by a unique number, namely the variance. For those fluctuations of $S(t)$ away from the most probable value $S_{pp}(t)$ given by (15) that are larger than the standard deviation of the Gaussian approximation (14), the average return and the variance are no more enough to quantify the whole spectrum of risks. This problem is very important because, as already pointed out, there is not a unique optimal investment strategy but rather a set of strategies each corresponding to a given risk aversion. Since risk aversion relies on a subtle combination of psychological and financial considerations, it is very important to provide a reliable appreciation of all the possible dimensions of the risks.

Technically, an answer is provided by the theory of large deviations, which establishes the probability, for large t , for the mean of independent and identically distributed random variables :

$$Prob[\frac{1}{t} \sum_{j=1}^t m_j \simeq x] \sim e^{ts(x)} , \quad (27)$$

where $s(x)$ is the Cramér function [10, 11, 12]. When applied to the binomial law (11) where the sum is of the type $\sum_{i=1}^t x_i$ with $x_i = 1$ with probability p and $x_i = 0$ with probability $1 - p$, this result is nothing but the one obtained from the Stirling approximation of $\binom{t}{j}$:

$$\mathcal{P}(j = xt) = e^{ts(x)} , \quad (28)$$

with

$$s(x) = x \log p + (1-x) \log(1-p) - x \log x - (1-x) \log(1-x) , \quad \text{for } 0 < x < 1 \quad (29)$$

$$s(x) = -\infty \quad \text{otherwise} . \quad (30)$$

Notice that $s(x)$ reduces to $s(x) = -\frac{(x-p)^2}{2p(1-p)}$ in the neighborhood of its maximum $x = p$, which retrieved exactly the expression (12) used until now. Figure 5 shows $s(x)$ and its parabolic approximation for $p = 0.5$ and $p = 0.95$. In the first case $p = 0.5$, the parabolic approximation, leading to the Gaussian distribution and to all the previously quoted results, overestimates the risks. For instance, the risk to loose systematically at each time step corresponds to $x = 0$, i.e. $\mathcal{P}(j = 0) = e^{ts(0)}$, where $s(0) \simeq -0.5$ according to the parabolic approximation, which can be compared to the exact value -0.7 . The probability to loose at each time step during t is thus

estimated as $e^{-0.5t} \approx 7 \cdot 10^{-3}$ and $e^{-0.7t} \approx 10^{-3}$ respectively for $t = 10$, leading to an overestimation of this risk by a factor 7 in the Gaussian approximation. For $p > 0.7$, the effect goes in the other direction. For instance, for $p = 0.95$, we see in the figure 5b that the function $s(x)$ is significantly above its parabolic approximation. As a consequence, large deviations are much more probable than predicted by the Gaussian approximation and the standard mean-variance theory.

In order to quantify this result, we express (10) as

$$S_x(t) = \left((1-f)^{1-x} [1 + (\lambda-1)f]^x \right)^t, \quad (31)$$

using $j = xt$. The distribution $\mathcal{P}(x)$ is given by (28) with (29). $\mathcal{P}(x)$ with the expression (31) provides the distribution of $S(t)$ by eliminating x . We can also use these two formulas by using x as a parameter: for $S(t)$ given, we determine the corresponding x from (31) which is then reported into (28) with (29) to derive the probability. Recall that x is simply the frequency of gains. Large deviations correspond to a frequency x different from p by an amount of order one.

As an illustration, let us take $p = 0.95$ and $\lambda = 1.1$, for whose realistic behaviors are found (5% probability to loose and a gain of 10%). Let us assume that the investor chooses $f = 0.4$ (which is close to the maximum) which corresponds to an average return $r(f = 0.4) = 1.2\%$. This result is also obtained from (31) with $x = p = 0.95$, using the definition $S(t) = e^{rt}$. What is the probability that, instead of getting the expected return, this investment looses on average -1.6% ? This value corresponds to $x = 0.9$ as seen from (31). The calculation of $s(x = 0.9)$ gives -0.0264 in the parabolic approximation and -0.0206 with the exact expression (29). The probability of this bad luck is thus $e^{-0.0264t}$ in the Gaussian approximation and $e^{-0.0206t}$ according to the Cramér expression. For $t = 100\tau$, we find respectively a probability equal to 7% and 13%. Thus, the Gaussian approximation under-estimates by almost a factor of two the probability of such a scenario.

Consider another example: what is the probability that the average return be equal to -0.5% ? Following the previous steps, we find that this return corresponds to $x \simeq 0.92$ for which $s(x = 0.92) = -9.5 \cdot 10^{-3}$ in the parabolic approximation and $s(x = 0.92) = -8.1 \cdot 10^{-3}$ for the exact expression (29). For $t = 1000\tau$, we obtain $7.5 \cdot 10^{-5}$ and $3 \cdot 10^{-3}$ respectively for the probability to get such a return. Here, the

Gaussian approximation is off by a factor 40!

To sum up the section, it is not possible in general to capture the degree of risk to which one is exposed to by the single variance parameter. A natural extension is to consider the full Cramér function as the quantity encoding the risk. The choice of the best investment becomes then even more dependent on additional considerations involving the aversion to rare and relatively large risks.

Is it possible to choose an investment that minimize the large risks? In the present simple model, the answer is: choose $f = 0$ if you are afraid of large risks. To see it, we pose the problem in the following way: we try to minimize the probability that the return be less than some level ρ , whose specific value is investor-dependent and not specified as long as it is sufficiently in the tail. This level ρ of a minimum tolerable return corresponds to a frequency $x_\rho(f)$ by the equation

$$(1 - f)^{1-x_\rho} [1 + (\lambda - 1)f]^{x_\rho} = e^\rho. \quad (32)$$

With (16), it gives x_ρ as

$$x_\rho = \sqrt{p(1-p)} \left(\frac{\rho - \log(1-f)}{\sigma(f)} \right), \quad (33)$$

where $\sigma(f) = \sqrt{v(f)}$ is the standard deviation. The worst return is of course $\log(1-f)$ for which x is zero. Notice that the value $x_\rho = p$ corresponds to the typical return $r(f)$ given by (17). For $\rho < r(f)$, the frequency of gains x_ρ is less than p .

The probability to observe a return less or equal to ρ is $\int_0^{x_\rho} P(x) dx \equiv \sum_{j=0}^{tx_\rho} \binom{t}{j} p^j (1-p)^{t-j}$, in which each term behaves like $e^{ts(x=j/t)}$. Is there an optimal f such that $\int_0^{x_\rho} e^{ts(x)} dx$ is minimum? Using (29), the extremum of $\int_0^{x_\rho} e^{ts(x)} dx$ with respect to f is given by the condition $0 = \frac{dx_\rho}{df} = -\frac{1-x_\rho}{1-f} + \frac{(\lambda-1)x_\rho}{1+(\lambda-1)f}$. However, this corresponds to a maximum since the second derivative is negative. The value f_1 which makes vanish $\frac{dx_\rho}{df}$ corresponds to a maximum of probability to gain a severe loss. These values are thus to be avoided. With (33), they are $1-f = \frac{\lambda}{\lambda-1}(1-x_\rho)$ with $\rho = \log \frac{\lambda}{\lambda-1} + (1-x_\rho) \log(1-x_\rho) + x_\rho \log x_\rho + x_\rho \log(\lambda-1)$. $(1-x_\rho) \log(1-x_\rho) + x_\rho \log x_\rho$ being negative with a minimum $-\log 2$ for $x_\rho = 1/2$ and vanishing for $x_\rho = 0$ and 1, there are non-trivial solutions for $x_\rho < p$. There exists other situations where an optimal investment can be devised such as to minimize the probability of large losses, as we discuss below.

3 Portfolios made of uncorrelated assets

Consider the sum

$$S(t) = \sum_{i=1}^N p_i x_i(t) , \quad (34)$$

where $x_i(t)$ represents the value at time t of the i -th asset and p_i is its weight in the portfolio. By normalization, we have $\sum_{i=1}^N p_i = 1$. $S(t)$ is the total value of the portfolio made of N assets at time t .

3.1 Gaussian limit and measure of risk by the variance

We would like to characterize the probability distribution $P_S(S(t))$ of $S(t)$, knowing the distributions $P_i(x_i)$ of the x_i for the different assets. For uncorrelated assets, the general formal solution reads

$$P_S(S) = \int dx_1 P_1(x_1) \int dx_2 P_2(x_2) \dots \int dx_j P_j(x_j) \dots \int dx_N P_N(x_N) \delta\left(S(t) - \sum_{i=1}^N p_i x_i(t)\right) . \quad (35)$$

Taking the Fourier transform $\hat{P}_S(k) \equiv \int_{-\infty}^{+\infty} dS P_S(S) e^{-ikS}$ of (35) gives, by the definition of the characteristic function [13],

$$\hat{P}_S(k) = \hat{P}_1(p_1 k) \hat{P}_2(p_2 k) \dots \hat{P}_j(p_j k) \dots \hat{P}_N(p_N k) . \quad (36)$$

This equation expresses that the Fourier transform of $P_S(S)$ is the product of the Fourier transform of the distribution of each constituent asset with an argument proportional to their respective weight in the portfolio. We now use the cumulant expansion of the characteristic function :

$$\log \hat{P}_j(p_j k) = -ik p_j C_1(x_j) - \frac{1}{2} p_j^2 k^2 C_2(x_j) + \mathcal{O}(p_j^3 k^3) , \quad (37)$$

where $C_1(x_j) \equiv \langle x_j \rangle$ is the average of x_j and $C_2(x_j) \equiv \langle x_j^2 \rangle - \langle x_j \rangle^2$ is the variance of x_j . We assume for the time being that these cumulants exist and are finite. The notation $\mathcal{O}(p^3 k^3)$ means that the higher order terms are at least of order $p^3 k^3$. In the limit where the number N of assets is very large, the weights p_i are of order $\frac{1}{N}$ (we avoid the special cases where one or a few assets are predominant). In this case, the previous cumulant expansion is warranted since all arguments are small and, keeping

terms up to the quadratic order, we obtain

$$\hat{P}_S(k) \simeq \exp\left[-ik \sum_{j=1}^N p_j C_1(x_j) - \frac{1}{2}k^2 \sum_{j=1}^N p_j^2 C_2(x_j)\right] . \quad (38)$$

Its Fourier inverse is

$$P_S(S) \simeq \frac{1}{\sqrt{2\pi V}} \exp\left[-\frac{(S - \langle S \rangle)^2}{2V}\right] , \quad (39)$$

where the average of S is

$$\langle S \rangle = \sum_{j=1}^N p_j \langle x_j \rangle , \quad (40)$$

et la variance V de S s'écrit

$$V = \sum_{j=1}^N p_j^2 C_2(x_j) . \quad (41)$$

These expressions express again the validity of the central limit theorem.

3.2 Large risks : the case of exponentially distributed assets

The previous Gaussian approximation (39) does not apply for large risks. We have already met this fact in the binomial model with the difference between (14) and (28, 29). We now present a slightly more general illustration of this fact by studying the case of assets that are exponentially distributed. It turns out that this situation is not far from reality when one deals with daily returns [14].

3.2.1 Determination of the distribution of portfolio values

To simplify, we assume that the distributions of asset return is symmetric :

$$P_j(\delta x_j) = \frac{1}{2} \alpha_j e^{\alpha_j |\delta x_j|} , \quad (42)$$

for $-\infty < \delta x_j < +\infty$, where $\langle \delta x_j \rangle = 0$ and the variance δx_j is $C_2(v_j) = \frac{1}{\alpha_j^2}$.

The Fourier transform of (42) is

$$\hat{P}_j(k) = \frac{1}{1 + (k\alpha_j^{-1})^2} . \quad (43)$$

Inserting this expression in (36) and taking the inverse Fourier transform, we get

$$P_S(\delta S) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{e^{ik\delta S}}{\prod_{j=1}^N [1 + (kp_j\alpha_j^{-1})^2]} . \quad (44)$$

We retrieve the Gaussian approximation by noting that

$$\frac{1}{1 + (kp_j\alpha_j^{-1})^2} = \exp\left(-\log[1 + (kp_j\alpha_j^{-1})^2]\right) \simeq \exp\left(-(kp_j\alpha_j^{-1})^2\right), \quad (45)$$

and thus

$$P_S(\delta S) \simeq \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik\delta S} \exp\left(-k^2 \sum_{j=1}^N (p_j\alpha_j^{-1})^2\right), \quad (46)$$

which recovers (39).

The integral (44) can be performed exactly by using Cauchy's theorem: one replaces the integral over the interval $-\infty < k < +\infty$ by an integral over a contour in the complex k plane. This contour is formed of the real axis and closes itself by a half-circle of infinite radius in the top half-plane. Cauchy's residue theorem then gives

$$P_S(\delta S) = \frac{1}{4} \sum_{j=1}^N \frac{1}{p_j\alpha_j^{-1}} \frac{1}{\prod_{i \neq j} \left(\left(\frac{p_j\alpha_j^{-1}}{p_i\alpha_i^{-1}} \right)^2 - 1 \right)} \exp\left[-\frac{|\delta S|}{p_j\alpha_j^{-1}}\right]. \quad (47)$$

This result is correct only if all the values $p_i\alpha_i$ are different. The special cases where several values are the same do not pose particular problems and can also be explicitly treated. This expression (47) shows that the large risks are given by

$$P_S(\delta S)_{\delta S \rightarrow -\infty} \simeq \frac{\hat{\alpha}}{4 \prod_{i \neq j_{max}} \left(\left(\frac{p_{j_{max}}\alpha_{j_{max}}^{-1}}{p_i\alpha_i^{-1}} \right)^2 - 1 \right)} e^{-\hat{\alpha}|\delta S|}, \quad (48)$$

where

$$\hat{\alpha} = \frac{\alpha_{j_{max}}}{p_{j_{max}}} \quad (49)$$

with j_{max} being the value of j which corresponds to the largest $p_j\alpha_j^{-1}$. The order of magnitude of the largest fluctuations of δS is thus $\hat{\alpha}^{-1}$, i.e. $\hat{\alpha}^{-1}$ provides a good estimation of the extreme risk of the portfolio.

3.2.2 Diversification in the presence of extreme risks

To protect oneself against the large risks, one should thus minimize $\hat{\alpha}^{-1}$, i.e. solve the optimization problem on the weights p_1, p_2, \dots, p_N , which consists first, for fixed weight p_1, p_2, \dots, p_N , in finding the smallest ratio $\frac{\alpha_j}{p_j}$ that we note $Min_{j=1, \dots, N} \frac{\alpha_j}{p_j}$, and

then to determine the weights p_1, p_2, \dots, p_N such that the smallest ratio be the largest possible. To sum up, we thus search for the solution of the max-min problem

$$\text{Max}_{p_1, p_2, \dots, p_N} \text{Min}_{(j=1, \dots, N)} \frac{\alpha_j}{p_j} . \quad (50)$$

To solve this problem, we invoke the following identity

$$\text{Min}_{(j=1, \dots, N)} \frac{\alpha_j}{p_j} = \lim_{q \rightarrow +\infty} \left(\sum_{j=1}^N \left(\frac{\alpha_j}{p_j} \right)^{-q} \right)^{-1/q} . \quad (51)$$

The idea is to invert the limit $q \rightarrow +\infty$ and the maximization with respect to the weights. At q fixed, we then maximize this expression with respect to the weights p_j together with the normalization constraint $\sum_{j=1}^N p_j = 1$, by using Lagrange multiplier method. The solution is $(\frac{\alpha_j}{p_j})^{-q} = \text{constant}$, independently of j . Then taking the limit $q \rightarrow +\infty$, the only possible solution is $p_j \propto \alpha_j$, which gives

$$p_k = \frac{\alpha_k}{\sum_{j=1}^N \alpha_j} , \quad (52)$$

after normalization. We will derive again this result below (see the equation (102)) by a different argument in terms of the minimization of the portfolio kurtosis, thus providing a different perspective to this result which constitutes the large deviation correction to the mean-variance approach for exponentially distributed assets. The same results have been derived independently in Ref.[15].

3.2.3 Large risks and optimal portfolios

A better measure of the risk than the sole knowledge of \hat{a} is provided by the probability that the loss δS of the portfolio be larger than some threshold λ . The parameter λ is a priori arbitrary and is chosen by the investor in view of his own risk aversion. λ is a so-called VaR, or value-at-risk. It is the value that can be lost at the probability level $\int_{-\infty}^{-\lambda} P_S(\delta S) dv$:

$$\int_{-\infty}^{-\lambda} P_S(\delta S) d\delta S = \frac{1}{4} \sum_{j=1}^N \frac{1}{\prod_{i \neq j} \left(\left(\frac{p_j \alpha_j^{-1}}{p_i \alpha_i^{-1}} \right)^2 - 1 \right)} \exp \left[-\frac{\lambda}{p_j \alpha_j^{-1}} \right]. \quad (53)$$

This expression provides the probability to observe a loss larger than λ during the unit time step considered here. Reciprocally, we can choose the confidence interval,

say 95% and determine what is the threshold λ such that the maximum loss remains smaller than λ in 95% out of all scenarios. The VaR λ is solution of

$$\int_{-\infty}^{-\lambda} P_S(\delta S) d\delta S = 0.05, \quad (54)$$

which can be explicitly solved using (53).

Let us now determine the weights of the portfolio that minimize the probability of loss larger than λ . The expression (53) is not very convenient for an analytical calculation. We thus turn to a more robust measure of the VaR given by the fact that a smooth estimation of $\int_{-\infty}^{-\lambda} P_S(\delta S) d\delta S$ is provided by $\int_{-\infty}^0 P_S(\delta S) (1 - e^{\frac{\delta S}{\lambda}}) d\delta S$. Indeed, $1 - e^{\frac{\delta S}{\lambda}}$ is close to 1 for $\delta S < -\lambda$ and is negligible in the other case. $(1 - e^{\frac{\delta S}{\lambda}})$ plays a role similar to a utility function, quantifying the sensitivity of the investor to large fluctuations. This leads to the following smooth definition of the Var λ :

$$TL_{\beta=0}(P_S(|\delta S|)) - TL_{\beta=\frac{1}{\lambda}}(P_S(|\delta S|)) = 1 - p, \quad (55)$$

where $TL_{\beta}(f(x)) \equiv \int_0^{\infty} e^{-\beta x} f(x) dx$ is the Laplace transform of the function $f(x)$. The first term $TL_{\beta=0}(P_S(|\delta S|))$ is nothing but the total probability for a loss to occur (irrespective of its amplitude). This expression (55) has two advantages: i) a more progressive interpolation of the losses to determine the VaR and ii) the use of the Laplace transform which can be directly estimated in the case of portfolios from the product of individual Laplace transforms of the distribution of each asset.

Let us change variable and write $p_j \equiv \rho_j^2$, where the ρ_j are the novel parameters over which to minimize. This change of variables ensures that the weights remain positive. Notice that we could relax this constraint and allow for negative weights which would correspond to so-called “short” positions. We thus would like to minimize

$$TL_{\beta=0}(P_S(|\delta S|)) - TL_{\beta=\frac{1}{\lambda}}(P_S(|\delta S|)) - \gamma \sum_{j=1}^N \rho_j^2 \quad (56)$$

with respect to the $\rho_1, \rho_2, \dots, \rho_N$. γ is a Lagrange parameter that ensures the normalization of the weights. Using the analytic expression of the Laplace transforms, we can write (56) as

$$1 - p(\lambda) - \gamma \sum_{j=1}^N \rho_j^2, \quad (57)$$

where

$$p(\lambda) \equiv \frac{\prod_{j=1}^N \alpha_j \lambda}{\prod_{j=1}^N (\alpha_j \lambda + \rho_j^2)} . \quad (58)$$

Putting to zero the derivative of (57) with respect to each ρ_k gives

$$p_k = \frac{1}{N} + \frac{\lambda}{N} \sum_{j=1}^N (\alpha_j - \alpha_k) . \quad (59)$$

This solution exists provided that λ is not too large such that the p_k 's remain positive. γ is eliminated by the normalization.

For large λ ,

$$1 - p(\lambda) \approx \sum_{j=1}^N \frac{p_j}{\alpha_j} , \quad (60)$$

which recovers the situation treated in section 3.2.2.

3.3 Beyond the Gaussian limit: cumulant expansion and large deviation theory

For arbitrary assets with finite variance, the VaR at the probability level p is given by

$$\int_{-\infty}^{-\lambda} P_S(\delta S) d\delta S = 1 - p . \quad (61)$$

The large deviation theorem allows us to get

$$P_S(\delta S) \sim e^{Ns(\delta S)} , \quad (62)$$

where the Cramér function $s(x)$ can be expressed in terms of all the distributions $P_j(\delta x_j)$ for $j = 1$ to N [10]

$$s(\delta S) = \text{Inf}_\beta \left(\frac{1}{N} \sum_{j=1}^N \log \hat{P}_j(p_j \beta) + \beta \delta S \right) . \quad (63)$$

Inf_β expresses the fact that we evaluate the term within the parenthesis for the value of β which minimizes it. This expression (63) together with (62) gives the distribution of deviations that can take arbitrarily large values, i.e. much beyond the Gaussian approximation.

This expression also contains the Gaussian limit valid for small fluctuations. This can be seen from the formula (37) adapted to the Laplace transform (by replacing ik

by β). Noting $\langle \delta S \rangle = \sum_{j=1}^N p_j C_1(\delta x_j)$ and $V = \sum_{j=1}^N p_j^2 C_2(\delta x_j)$ (equations (40, 41)), the expression (63) becomes

$$Ns(\delta S) = \text{Inf}_\beta(-\beta\langle\delta S\rangle + \frac{1}{2}V\beta^2 + \beta\delta S) . \quad (64)$$

$s(\delta S)$ can then be obtained as the solution of a simple quadratic minimization. The value of β that minimizes the expression within the parenthesis is

$$\beta = -\frac{\delta S - \langle\delta S\rangle}{V} . \quad (65)$$

We thus finally obtain

$$s(\delta S) = -\frac{(\delta S - \langle\delta S\rangle)^2}{2NV} . \quad (66)$$

Reporting in (62), we thus retrieve the Gaussian approximation.

Let us now use (63) to express the first leading corrections to the Gaussian approximation (39). In this goal, we expand $\log \hat{P}_j(p_j\beta)$ on the cumulants c_n^j :

$$\log \hat{P}_j(\beta) = \sum_{n=1}^{\infty} \frac{c_n^j}{n!} (-\beta)^n . \quad (67)$$

Interchanging the sums over the assets j and over the cumulants n , we write (63) as

$$Ns(\delta S) = \text{Inf}_\beta \left(\beta(S - \langle S \rangle) + \frac{1}{2}V\beta^2 - \frac{1}{6}C_3\beta^3 + \frac{1}{24}C_4\beta^4 + \dots \right) , \quad (68)$$

where $\langle S \rangle$ and V are given by (40) and (41),

$$C_3 = \sum_{j=1}^N p_j^3 c_3^j , \quad (69)$$

and

$$C_4 = \sum_{j=1}^N p_j^4 c_4^j . \quad (70)$$

If the distributions of the price variations of the assets are non-symmetric, then $C_3 \neq 0$ and the first correction to the Gaussian approximation reads

$$P_S(S) \simeq \exp\left[-\frac{(S - \langle S \rangle)^2}{2V} \left(1 - \frac{C_3(S - \langle S \rangle)}{3V^2}\right)\right] . \quad (71)$$

For symmetric distributions such that $c_3^j = 0$, the leading correction is proportional to the kurtosis $\kappa = \frac{C_4}{V^2}$:

$$P_S(S) \simeq \exp\left[-\frac{(S - \langle S \rangle)^2}{2V} \left(1 - \frac{5C_4(S - \langle S \rangle)^2}{12V^3}\right)\right] . \quad (72)$$

For a typical fluctuation $S - \langle S \rangle \sim \sqrt{V}$, the relative size of the correction is order $\frac{5C_4}{12V^2} = \frac{5\kappa}{12}$. Notice the negative sign of the correction proportional to C_4 which means that large deviations are more probables than extrapolated by the Gaussian approximation.

Let us illustrate these results for the exponential distributions. Consider the simple case where all assets have the same distribution, i.e. $\alpha_j = \alpha$ for all j 's. Let us also take all weights p_j equal to $1/N$. The expression (63) then yields

$$s(\delta S) = \log(\alpha \delta S) + 1 - \alpha \delta S. \quad (73)$$

We recover directly this result by using the exact relation (44). Indeed, the family of Gamma functions is close under convolution ([16], p. 47). Applying this result to the variable $\delta S = \frac{1}{N} \sum_{j=1}^N \delta x_j$, this explains the additional factor N in the exponential and the prefactor. We thus have

$$P_S(\delta S) = \frac{N\alpha}{\Gamma(N)} (N\alpha \delta S)^{N-1} e^{-N\alpha \delta S}. \quad (74)$$

The expression (73) reported in (62) yields indeed (74). This result shows that the extreme tail of the distribution remains essentially of the exponential form $e^{-N\alpha \delta S}$, in agreement with our previous results.

The figure 6 illustrates these results by showing the function $s(y \equiv \alpha \delta S) = \log(y) + 1 - y$, in comparison to its parabolic approximation $s_g(y) = -\frac{1}{2}(y - 1)^2$. It is clear that the parabolic approximation is correct only for small deviations around the mean value $y = 1$. For $|y| = 3$, $s_g(y)$ is already twice as large as $s(y)$, leading to a severe under-estimation of large fluctuations by the Gaussian approximation. Notice also that for large y , the function $s(y)$ becomes essentially parallel to the straight line of slope -1 , thus justifying the asymptotic shape (62) of the distribution tail.

4 General analysis of portfolios

Let us consider N assets X^a , $a = 1, \dots, N$, each of which has an instantaneous value x_k^a at time k . We now present the analysis of portfolio optimization in the presence of large fluctuations and for the general case of correlated assets. Before reaching this goal, we first recall the standard mean-variance approach valid within the Gaussian approximation.

4.1 Correlation matrix between assets

Within the Gaussian approximation, correlations between assets can be fully characterized by the determination of the correlation matrix

$$V_{ab} \equiv \langle \delta x^a \delta x^b \rangle, \quad (75)$$

which also reads $V \equiv \langle X^a X^{aT} \rangle$ in matrix notation. X^a is the column vector with row element x^a , for $a = 1$ to N and the exponent T stands for the transpose operation. By construction, V_{ab} is symmetric and for non-singular cases can be diagonalized with all its eigenvalues λ_k being real. The elements of the eigen-vectors P_α are also real [17]. Notice that V_{ab} is a symmetric matrix with only N and not $N(N+1)/2$ independent degrees of freedom. There is thus an orthogonal matrix A such that $A^T V A = \lambda^d$, where λ^d is the diagonal matrix made with the eigenvalues of V . Recall that A is orthogonal if $A^{-1} = A^T$, i.e. $\sum_c A_{ac} A_{bc} = \delta_{ab}$. Then the vector $U^a \equiv A^T X^a$ has the following correlation matrix $\langle A^T X^a (A^T X^a)^T \rangle = A^T \langle X^a X^{aT} \rangle A = A^T V A = \lambda^d$, i.e. U^a has all its components that are uncorrelated with each other. The quadratic average $\langle (U^a)^2 \rangle$ of an element of this vector is then equal to the corresponding eigenvalue λ_a . We can thus decompose the assets x^a over the set of independent factors U^a , in matrix form $X^a = A U_a$ and explicitly

$$x_k^a = \sum_{b=1}^N A_{ab} u_k^b. \quad (76)$$

Since the U^a are independent and of variance λ , the correlation matrix $\langle X^a X^{aT} \rangle$ is equal to $A \lambda^d A^T$, i.e.

$$\langle \delta x^a \delta x^b \rangle = (A \lambda^d A^T)_{ab} = \sum_c \lambda_c A_{ac} A_{bc}. \quad (77)$$

This decomposition thus reduces the problem of correlated assets to the previous case of uncorrelated assets. Within the Gaussian approximation, the correlation matrix V determines completely the distribution of price variations δx^a through

$$P(X^a) \propto \exp\left(-\frac{1}{2} X^{aT} V^{-1} X^a\right). \quad (78)$$

We can verify (75) directly from (78) by making explicit the calculation of the correlation with the probability (78).

4.2 Optimal portfolio within the mean-variance approach

Consider a portfolio made of N assets with respective weight p_a , with $a = 1$ to N . The variation $\delta S(t)$ of the value of the portfolio during the time interval τ is

$$\delta S(t) = \sum_{a=1}^N p_a \delta x^a(t) \quad (79)$$

($= p^T X^a$ in matrix notation). The variance $\langle [\delta S(t)]^2 \rangle$ is

$$\langle [\delta S(t)]^2 \rangle = p^T V p . \quad (80)$$

Notice that we can retrieve this result from (78) by using the fact that the distribution of $\delta S(t)$ is formally $P(\delta S(t)) \propto \int dX^a P(X^a) \delta(S - \sum_{a=1}^N p_a \delta x^a(t))$. Its Fourier transform can be expressed under the form

$$\prod_{i=1}^N \left(\int dx_i \right) \exp \left(-\frac{1}{4} x_i A_{ij}^{-1} x_j + y_i x_i \right) = \sqrt{\frac{\pi^N}{\det(A_{ij})}} \exp(y_i A_{ij} y_j) , \quad (81)$$

which by inverse Fourier transform gives

$$P(\delta S(t)) \propto \exp \left(-\frac{[\delta S(t)]^2}{2p^T V p} \right) . \quad (82)$$

It is useful to express $\langle [\delta S(t)]^2 \rangle = p^T V p$ under a form that exploits the decomposition over the independent components U^a . Reporting $X^a = A U^a$ in $\delta S(t) = p^T X^a$, we obtain $\delta S(t) = \hat{p}^T U^a$, where

$$\hat{p} \equiv A^T p . \quad (83)$$

This expression represents the portfolio as made of a set of effective assets U^a , $a = 1$ to N which are uncorrelated, with effective weights \hat{p}^a . We can thus directly use the previous results as soon as we specify the U^a 's. We have

$$\langle [\delta S(t)]^2 \rangle = \hat{p}^T \lambda^d \hat{p} = \sum_{a=1}^N \lambda_a [\hat{p}^a]^2 . \quad (84)$$

The average return is given by

$$\langle \delta S(t) \rangle = \sum_{a=1}^N p_a \langle \delta x^a(t) \rangle = p^T \langle X^a \rangle = \hat{p}^T \langle U^a \rangle . \quad (85)$$

Within the Gaussian approximation, the risk associated with the portfolio defined in terms of the weights p^a is quantified by the variance $\langle [\delta S(t)]^2 \rangle - \langle \delta S(t) \rangle^2$ given from (84) by replacing V by the covariance matrix, i.e. by subtracting $\langle \delta x^a \rangle \langle \delta x^b \rangle$ to $\langle \delta x^a \delta x^b \rangle$. In the sequel, we use the same notation V for the covariance matrix (and the other derived ones).

The mean-variance approach developed by Markovitz, that we already visited in the first sections, consists in looking for the weights p^a such that the return be the largest possible for a given variance $\langle [\delta S(t)]^2 \rangle - \langle \delta S(t) \rangle^2$ or equivalently that the variance $\langle [\delta S(t)]^2 \rangle - \langle \delta S(t) \rangle^2$ be minimum for a given return. The solution is obtained by minimizing the following expression with respect to the weights p^a

$$M \equiv \sum_{j=1}^N \lambda_j [\hat{p}^j]^2 - \alpha_1 \sum_{j=1}^N \hat{p}^j \langle U^j \rangle - \alpha_2 \sum_{j=1}^N \hat{p}^j , \quad (86)$$

where $\alpha_{1,2}$ are the Lagrange parameters introduced to constraint the minimization at fixed gain $\langle \delta S(t) \rangle$ and to ensure the normalization of the weights.

The extremalization with respect to a weight p^a gives

$$\frac{\partial M}{\partial p^a} = \sum_j \frac{\partial M}{\partial \hat{p}^j} \frac{\partial \hat{p}^j}{\partial p^a} = \sum_j \left(2\lambda_j \hat{p}^j - (\alpha_1 \langle U^j \rangle - \alpha_2) (A^T)_{ja} \right) , \quad (87)$$

which gives the following matrix equation

$$\lambda^d \hat{p} = \alpha_1 \langle U^a \rangle + \alpha_2 \vec{1} , \quad (88)$$

where $\vec{1}$ is the single column vecteur with unit elements. With (83), $\langle U^a \rangle = A^T \langle X^a \rangle$ and $V = A \lambda^d A^T$, we finally obtain

$$p^a = \alpha_1 V_{ab}^{-1} \langle X^b \rangle + \alpha_2 A_{ab} \vec{1}^b . \quad (89)$$

Imposing $\sum_{i=1}^N p^i = 1$ and $\sum_{i=1}^N p_i \langle X^i \rangle = \langle \delta S(t) \rangle$ yields α_1 and α_2 as functions of $\langle \delta S(t) \rangle$. Varying $\langle \delta S(t) \rangle$ enables us to derive the mean-variance curve, called the “efficient frontier”. α_1 can be interpreted as the risk aversion parameter. Numerous books discuss these solutions [18].

Notwithstanding a wide application due to its convenience and simplicity, the mean-variance approach hides several severe problems.

- Mean-variance portfolios are not very diversified and have the tendency to select assets with comparable risks.
- A frequent reallocation is called for to address the non-stationarity of the estimation of the covariance matrix.

We now expose several successive generalizations of the mean-variance approach that specifically address the limitation of the Gaussian approximation.

4.3 Quasi-Gaussian parametrization

It may be useful to attempt to represent the multivariate distribution of price variations of N assets by the following expression

$$P(X) = F\left((X - X^0)^T V^{-1}(X - X^0)\right). \quad (90)$$

X^0 is the unit column vector of the average of the price variations. The function F is kept a priori arbitrary. Notice that if F is an exponential, we retrieve the Gaussian distribution (78) and V becomes the covariance matrix. Consider a portfolio with weights given by the unit column vector p . Its value variation during the unit time is $\delta S(t) = \sum_{a=1}^N p_a \delta x^a(t) = p^T X$ using the matrix notation. The distribution $P(\delta S)d\delta S$ can be written as

$$P(\delta S) = \int dX F\left((X - X^0)^T V^{-1}(X - X^0)\right) \delta(\delta S - p^T X). \quad (91)$$

To estimate this integral, we isolate one of the assets $x_1 = x_1^0 + y_1$ and write, using $Y = X - X^0$,

$$Y^T V^{-1} Y = V_{11}^{-1} y_1^2 + 2(v^T y) y_1 + y^T \mathcal{V}^{-1} y, \quad (92)$$

where V_{ij}^{-1} is the element ij of the matrix V^{-1} , y is the unit column vector $(y_2, y_3, \dots, y_N)^T$ of dimension $N - 1$, v is the unit column vector $(V_{21}^{-1}, V_{31}^{-1}, \dots, V_{N1}^{-1})^T$ of dimension $N - 1$, and \mathcal{V}^{-1} is the square matrix of dimension $N - 1$ by $N - 1$ derived from V^{-1} by removing the first row and first column. The factor 2 in $2(v^T y) y_1$ comes from the symmetric structure of the matrix V^{-1} .

We can now express the condition $\delta(\delta S - p^T X)$ in the integral (91):

$$\frac{1}{p_1} \delta(y_1 - \frac{1}{p_1} (\delta S - p^T X_0 - \mathcal{P}^T y)) , \quad (93)$$

where \mathcal{P} is the unit column vector $(p_2, p_3, \dots, p_N)^T$ of dimension $N-1$. The integration over the variable y_1 cancels out the Dirac function and we obtain the argument of the function F under a quadratic form in the variables S et y . Using the identity

$$X^T V^{-1} X + X^T Y = \hat{X}^T V^{-1} \hat{X} - \frac{1}{4} Y^T V Y, \quad (94)$$

where $\hat{X} = X + VY$, we obtain

$$P(\delta S) = \int d\hat{y} F\left(\hat{y}^T M^{-1} \hat{y} + \frac{\delta S^2}{P^T V P}\right), \quad (95)$$

where the integral is carried out over the space of vectors \hat{y} of dimension $N-1$ and

$$M^{-1} \equiv \mathcal{V}^{-1} - \frac{2}{p_1} \left(v - \frac{V_{11}^{-1}}{2p_1} \mathcal{P}\right) \mathcal{P}^T. \quad (96)$$

We can finally write

$$P(\delta S) = \mathcal{F}\left(\frac{\delta S^2}{p^T V p}\right), \quad (97)$$

where $\mathcal{F}(x)$ is defined by (95). The prefactor $\frac{1}{P^T V P}$ of δS^2 is simply deduced by remarking that it is independent of the function f and thus equal to that obtained for the Gaussian case.

This remarkable result shows that the typical amplitude of the fluctuations of the values of the portfolio is still controlled by the quasi-variance $p^T V p$ defined as in the Gaussian case (82). It is then natural to optimize the portfolio using the quasi-variance as the measure of the risk. This parametrization provides a natural generalization of the standard mean-variance Markovitz approach. There is however a danger for the largest fluctuations in relying only on this insight. Indeed, the expression (95) for $\mathcal{F}(x)$ shows that the explicit dependence of the distribution $P(\delta S)$ is in general a function of the asset weights constituting the portfolio, beyond the simple dependence in $p^T V p$. Minimizing only $p^T V p$ may thus be insufficient because it may be linked to a dangerous deformation of $\mathcal{F}(x)$ in the tail.

4.4 Generalization to non-gaussian correlated assets

We assume that it is still possible to decompose the assets X^a on a set of effective independent assets U^a :

$$X^a = A U^a. \quad (98)$$

The unit column vector U^a can be interpreted as the set of “explanatory” factors of the price variations. For this decomposition to be useful, the matrix A should be constant while X^a and U^a fluctuate in time. This provides a set of orthogonal assets whose fluctuations are uncorrelated. An estimation of the matrix A can be obtained from the covariance matrix as above. A first generalization consists in relaxing the condition that the U^a be distributed according to a Gaussian distribution as in the previous section. From the data X^a and the construction of the matrix A , we can study each effective asset U^a and determine its distribution $P_a^U(U^a)$. If the decomposition works and the effective assets are not correlated, we can write for each effective asset U^a the cumulant expansion of the Laplace transform of $P_a^U(U^a)$ under the form (67). Since the variation of the portfolio value is given as before by $\delta S(t) = p^T X^a = \hat{p}^T U^a$, we obtain the Cramér function of the distribution of $\delta S(t)$ under the form (68) with (69,70) where the p_j ’s are replaced by $\hat{p}^j = A^T p$ according to the equation (83). The cumulants corresponds to the effective assets U^a . We thus obtain the distribution of $\delta S(t)$ which accounts for the first leptokurtic corrections given by (71) for the non-symmetric case and by (72) for the symmetric case.

Let us consider the symmetric case (72). This expression shows that the risk is not uniquely represented by the variance V but also by the coefficient C_4 as well as by all higher order cumulants. There is a danger in working only on the variance because minimizing only the variance may lead to larger fluctuations than before the minimization! To see this, we note that typical fluctuations $S - \langle S \rangle \sim V$ have a probability becoming much larger than the Gaussian estimate because, if V becomes small by the action of the naive mean-variance optimization, when C_4 does not decrease in proportion, the correction term $\frac{C_4}{V}$ remains large. At this order in the perturbative expansion, it is necessary to take into account non only V but also C_4 . The minimization of V must be performed by controlling and even maximizing simultaneously the ratio $\frac{V^2}{C_4}$, this at fixed return. The second condition ensures that the weight of the non-Gaussian tail remains small, thus mastering the large risks.

This optimization problem has no unique solution due to the existence of several contradictory constraints. In standard economic theory, one relies on the use of utility function to decide the relative importance of the different constraints. For the sake

of illustration, let us propose the following strategies.

- Minimize V while maximizing $\frac{V^2}{C_4}$ can be obtained by minimizing the ratio

$$V/(\frac{V^2}{C_4})^\beta = V^{1-2\beta} C_4^\beta \quad \text{with} \quad 0 \leq \beta < \frac{1}{2}, \quad (99)$$

to ensure that both V and C_4 are simultaneously decreased. β quantifies the relative weight given to the kurtosis.

- One can introduce a parameter ψ which quantifies the sensitivity of the investor with respect to large fluctuations measured by C_4 and minimize the function

$$V - \psi \frac{V^2}{C_4} - \alpha_1 R - \alpha_2 \sum_j p_j, \quad (100)$$

where α_1 ensures a fixed return and α_2 accounts for the normalization of the asset weights in the portfolio.

4.5 Exponential distributions

We have already analyzed the problem of a portfolio constituted of assets with exponentially distributed price variations. Let us reexamine this problem within the framework of the cumulant expansion. We work directly on the effective explanatory assets U^a that are non-correlated.

The distribution $P_S(\delta S)$ is given by (44) with the p_j 's replaced by \hat{p}_j (in the sequel, we omit the hat). The coefficients α_j are the exponents of the exponential distributions of price variations of the independent assets. An expansion in power of k^2 as in (45) but up to the order k^4 gives

$$P_S(\delta S) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik\delta S} \exp\left(-k^2 \sum_{j=1}^N (p_j \alpha_j^{-1})^2 + \frac{k^4}{2} \sum_{j=1}^N (p_j \alpha_j^{-1})^4 + \dots\right). \quad (101)$$

$P_S(\delta S)$ is thus of the form (71) with $V = 2 \sum_{j=1}^N (p_j \alpha_j^{-1})^2$ and $C_4 = \sum_{j=1}^N (p_j \alpha_j^{-1})^4$. Let us assume that all assets have the same return, fixed to zero without loss of generality. We thus focus our analysis to the minimization of the risk.

The first scenario consists according to the Gaussian approach to minimizing the variance $\frac{\sum_{j=1}^N (p_j \alpha_j^{-1})^2}{(\sum_{j=1}^N p_j)^2}$. The solution is

$$p_k^{(1)} = \frac{\alpha_k^2}{\sum_{j=1}^N \alpha_j^2}. \quad (102)$$

The second scenario consists in focusing on the large risks, here quantified by the ratio $\frac{V^2}{C_4}$ which describes the large fluctuations of the price variations. The ratio $\frac{V^2}{C_4}$ can be written as $\frac{(\sum_{j=1}^N W_j)^2}{\sum_{j=1}^N W_j^2}$ with $W_i \equiv (p_j \alpha_j^{-1})^2$. It is maximum when all the weights W_i are equal, i.e.

$$p_k^{(2)} = \frac{\alpha_k}{\sum_{j=1}^N \alpha_j} . \quad (103)$$

This is the result (52) already obtained from the condition of minimizing the probability of large losses.

Notice that the two strategies (102) and (103) become identical $p_k = 1/N$, if all assets have the same parameter α . Indeed, the large risks are diversified from the beginning since the probability tails are identical in this case. The strategy (103) corresponds to balance the risks equally over all assets. In contrast, the portfolio (102) over-controls the balance: if an asset exhibits a larger risk in the tail, i.e. its coefficient α is significantly smaller than the others, it is almost completely absent from the portfolio. If an asset has a large α , it will have a large weight in the portfolio and its parameter $\frac{\alpha}{p}$ becomes smaller than the other ones. We see here the general trend exhibited by solutions obtained from the mean-variance approach which correspond to a diversification only on assets presenting comparable risks, while excluding almost completely the more risky assets.

The strategy (103), which implies that $p_k^{(2)} \alpha_k^{-1} = \sum_{j=1}^N \alpha_j$ is a constant independent of k , leads to a simple exact expression of the distribution of the fluctuations δS of the portfolio value. Indeed, the formula (44) gives in a manner similar to (74):

$$P_S(\delta S) = \frac{\sum_{j=1}^N \alpha_j}{\Gamma(N)} \left(\left[\sum_{j=1}^N \alpha_j \right] \delta S \right)^{N-1} \exp \left(- \left[\sum_{j=1}^N \alpha_j \right] \delta S \right) . \quad (104)$$

This Gamma distribution converges in its center to the Gaussian law while keeping an exponential tail for the largest fluctuations, with an exponent $\sum_{j=1}^N \alpha_j$. This can be compared to the distribution (47) with the value of p_j 's given by (102):

$$P_S(\delta S) = \frac{1}{4} \sum_{j=1}^N \frac{\sum_{i=1}^N \alpha_i^2}{\alpha_j} \frac{1}{\prod_{i \neq j} \left(\left(\frac{\alpha_i}{\alpha_j} \right)^2 - 1 \right)} \exp \left(- \frac{\sum_{i=1}^N \alpha_i^2}{\alpha_j} |\delta S| \right) . \quad (105)$$

One sees clearly that, as expected, the large losses are better controlled by the strategy (103) yielding typical fluctuations of the order of $\delta S^{(2)} \sim [\sum_{j=1}^N \alpha_j]^{-1}$, while the

strategy (102) leads to typical fluctuations of order $\delta S^{(1)} \sim \frac{\alpha_{max}}{\sum_{i=1}^N \alpha_i^2}$, where α_{max} is the largest of the exponents. Let us assume for the sake of illustration that the exponents α are distributed according to a Gaussian law with mean $\langle \alpha \rangle$ and variance σ^2 , then $\delta S^{(2)} \sim \frac{\langle \alpha \rangle^{-1}}{N}$ and $\delta S^{(1)} \sim \frac{\langle \alpha \rangle^{-1}}{N} + \frac{\sigma}{\langle \alpha \rangle^2} \frac{\sqrt{\log N}}{N}$.

4.6 General theory and cumulant expansion from the generalized correlation matrices

4.6.1 Strategies

Let us consider the case where the assets are distributed according to the arbitrary joint distribution

$$P(x_1, x_2, \dots, x_i, \dots, x_N, t/x_1^0, x_2^0, \dots, x_i^0, \dots, x_N^0, 0), \quad (106)$$

which is in general non-Gaussian and with arbitrary inter-asset correlations. A natural strategy to determine a portfolio is to calculate the weights p_k of the assets in the portfolio, $k = 1$ to N , which minimize the risk of losses larger than a chosen VaR λ , this in the presence of other constraints for instance on the return :

$$\frac{\partial \left(\int_{-\infty}^{-\lambda} P_S(\delta S) d\delta S - \alpha_1 \sum_{j=1}^N p_j \langle \delta x_j \rangle - \alpha_2 \sum_{j=1}^N p_j \right)}{\partial p_k} = 0 \quad (107)$$

for $k = 1$ to N , where α_1 and α_2 are Lagrange parameters. λ becomes a parameter that quantifies the risk aversion of the investor. The residual probability $\mathcal{P}_{min}(\lambda)$ obtained after the optimization gives the mean frequency of the occurrence of losses equal or larger than λ . This approach requires a three-dimensional representation along the following axis :

1. the VaR λ controlling the acceptable level of loss,
2. the loss probability $\mathcal{P}(\lambda)$ at this level,
3. the expected return.

It generalizes the usual two-dimensional representation of the mean-variance Markovitz diagram. This is the price to pay when the distributions of asset price variations are not quantified by a single risk parameter, the variance. Note that

there is another situation where the risks can be captured by a single parameter, i.e. the case of power law distributions quantified by the scale parameter. This case has been discussed previously [19] and the general solution for the portfolio optimization has been given [19] as a rather straightforward generalization of the usual mean-variance approach.

In the same spirit as for the minimization of the probability of losses larger than VaR, let us mention the strategy consisting in maximizing the probability to obtain a minimum return $\int_{\delta S_{min}}^{\infty} P_S(\delta S) d\delta S$ [20]. Generalizing further, we can propose to minimize the weight $\int_{-\infty}^{-\lambda} P_S(\delta S) d\delta S$ of the losses in the presence of the constraint that the probability of a minimum gain is fixed, thus generalizing in the probability space the mean-variance concept. This view point is stimulated by the observation that the returns and losses aggregated over long period of times are often mainly caused by large amplitude price variations that occurred over a very tiny fraction of the total time of the investment. For instance, for the US *S&P500* index, from 1983 to 1992, 80% of the total return stems from 1.6% of the trading time. This leads to optimization problems similar to (107).

The most general approach consists in first determining the complete distribution $P(\delta S(t))$ of the price variations of the portfolio as a function of the distributions of the underlying assets. Once characterized, one knows fully the impact of the asset weights on the portfolio. The optimization of the portfolio can then proceed using suitable risk measures.

4.6.2 Distribution of the price variations of the portfolio

$P(\delta S(t))$ is formally given by

$$P(\delta S(t)) = \int dX^a P(X^a) \delta(S - \sum_{a=1}^N p_a \delta x^a(t)) , \quad (108)$$

where we use again the matrice notation

$$P(X^a) = P(\delta x_1, \delta x_2, \dots, \delta x_i, \dots \delta x_N) \quad (109)$$

and $\int dX^a$ stands for N integrals over the price variations of the N assets. The probability being positive, we can parametrize it without loss of generality

$$P(X^a) \equiv \exp\left(-\frac{1}{2} X^{aT} V^{-1} X^a - \mathcal{V}(X^a)\right), \quad (110)$$

where we distinguish a Gaussian part $\exp[-\frac{1}{2}X^{aT}V^{-1}X^a]$ and a residual term $\exp[-\mathcal{V}(X^a)]$ representing the non-Gaussian part. It may be dominant over the Gaussian part. The expression (110) defines the Gaussian correlation matrix V and the non-Gaussian contribution \mathcal{V} , which contains all terms with power strickly larger than 3 in X^a . With (108), we obtain the expression of the Fourier transform of $P(\delta S)$:

$$\hat{P}(k) = \int dX^a \exp\left(-\frac{1}{2}X^{aT}V^{-1}X^a - \mathcal{V}(X^a) + H^T X^a\right), \quad (111)$$

where $H \equiv ikp$ and p is as before the unit column vector of the asset weights in the portfolio.

The most general and powerful technique to determine $P(\delta S)$ from the exact expression (111) consists in using the “technology” of functional integrals [21]. The basic idea is to reduce the evaluation of the integrals to that of Gaussian integrals. The key technical remark is

$$\begin{aligned} \frac{\partial}{\partial H^c} \int dX^a \exp\left(-\frac{1}{2}X^{aT}V^{-1}X^a - \mathcal{V}(X^a) + H^T X^a\right) = \\ \int dX^a X^c \exp\left(-\frac{1}{2}X^{aT}V^{-1}X^a - \mathcal{V}(X^a) + H^T X^a\right). \end{aligned} \quad (112)$$

Expanding the exponentiel as a formal series, we use the identity (112) and then resum the series to obtain formally

$$\hat{P}(k) = \exp\left(-\mathcal{V}\left(\frac{\partial}{\partial H^c}\right)\right) \int dX^a \exp\left(-\frac{1}{2}X^{aT}V^{-1}X^a + H^T X^a\right). \quad (113)$$

The notation $\mathcal{V}(\frac{\partial}{\partial H^c})$ means that each component X^c is replaced by the operator $\frac{\partial}{\partial H^c}$. By this trick, we have transformed a non-Gaussian integral into an operator applied on a Gaussian integral that can be calculated explicetely by using (81). This yields

$$\hat{P}(k) = \exp\left(\sum_{c=1}^N \mathcal{V}\left(\frac{\partial}{\partial H^c}\right)\right) \exp\left(\frac{1}{2}H^{aT}VH^a\right). \quad (114)$$

Notice that we retrieve the Gaussian case for $\mathcal{V} = 0$, i.e. the Fourier transform of (82) by replacing H by ikp . By the normalization of probabilities, $\hat{P}(0) = 1$.

One can show [21] that $\hat{P}(k)$ given by (114) can be written as

$$\hat{P}(k) \equiv e^{W(H)}, \quad (115)$$

where $W(H)$ presents a systematic expansion

$$W(H) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j_1=1}^N \dots \sum_{j_n=1}^N H_{j_1} \dots H_{j_n} G_c^{(n)}(j_1, \dots, j_n) . \quad (116)$$

H_j for $j = 1, \dots, N$ is one of the components of the unit column vector ikp . The functions $G_c^{(n)}(j_1, \dots, j_n)$ can be expressed explicitly in terms of \mathcal{V} [21]. The usefulness of this formulation is that each H brings in a power of k :

$$\hat{P}(k) = \exp \left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \sum_{j_1=1}^N \dots \sum_{j_n=1}^N p_{j_1} \dots p_{j_n} G_c^{(i)}(j_1, \dots, j_n) \right) . \quad (117)$$

This expansion defines the cumulants $c_n \equiv (-i)^n \frac{d^n}{dk^n} \hat{P}(k)|_{k=0}$ of the distribution $P(\delta S)$:

$$c_n = \sum_{j_1=1}^N \dots \sum_{j_n=1}^N p_{j_1} \dots p_{j_n} G_c^{(i)}(j_1, \dots, j_n) . \quad (118)$$

Notice that the correlations between the different assets are taken into account in the functions $G_c^{(i)}(j_1, \dots, j_i)$. They play the role of generalized correlation functions which quantify the pairwise, triplets, etc, correlations (a multiplet can contain the same asset several times, thus capturing the effect of self-correlations).

One can also use another systematic expansion

$$\hat{P}(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \sum_{j_1=1}^N \dots \sum_{j_n=1}^N p_{j_1} \dots p_{j_n} G^{(i)}(j_1, \dots, j_n) , \quad (119)$$

where the functions $G^{(i)}(j_1, \dots, j_n)$ can be expressed in terms of the $G_c^{(i)}(j_1, \dots, j_n)$. The functions $G^{(i)}(j_1, \dots, j_n)$ are analogous to the moments which can be related to $G_c^{(i)}(j_1, \dots, j_n)$ which are analogous to the cumulants.

Let us pursue this analysis and consider the symmetric case where the first term in the expansion of $\mathcal{V}(X)$ is quartic:

$$\begin{aligned} \mathcal{V}(X) = & \sum_{a=1}^N \sum_{b=1}^N \sum_{c=1}^N \sum_{d=1}^N v_{abcd} X^a X^b X^c X^d + \\ & \sum_{a=1}^N \sum_{b=1}^N \sum_{c=1}^N \sum_{d=1}^N \sum_{e=1}^N \sum_{f=1}^N v_{abcdef} X^a X^b X^c X^d X^e X^f + \dots \end{aligned} \quad (120)$$

Keeping for the time being only the quartic terms proportional to v_{abcd} , we obtain

$$\mathcal{V}\left(\frac{\partial}{\partial H^c}\right) \exp\left(\frac{1}{2} H^{aT} V H^a\right) = \sum_{a=1}^N \sum_{b=1}^N \sum_{c=1}^N \sum_{d=1}^N v_{abcd} \left(V_{ab} V_{cd} + V_{ac} V_{bd} + V_{ad} V_{bc} + \right.$$

$$V_{ab}S_cS_d + V_{ac}S_bS_d + V_{ad}S_bS_c + V_{bc}S_aS_d + V_{bd}S_aS_c + V_{cd}S_aS_c + S_aS_bS_cS_d \Big) , \quad (121)$$

where $S_a \equiv ik \sum_{k=1}^N V_{ak}p_k$. We thus see all possible term combinations. In order to retrieve the fourth order cumulant, we notice that

$$\begin{aligned} & \exp \left[v_{abcd} \left(V_{ab}V_{cd} + V_{ac}V_{bd} + V_{ad}V_{bc} + \right. \right. \\ & \quad \left. \left. V_{ab}S_cS_d + V_{ac}S_bS_d + V_{ad}S_bS_c + V_{bc}S_aS_d + V_{bd}S_aS_c + V_{cd}S_aS_c + \right. \right. \\ & \quad \left. \left. \left(1 - \frac{1}{2}(V_{ab}V_{cd} + V_{ac}V_{bd} + V_{ad}V_{bc}) \right) S_aS_bS_cS_d \right) \right] \end{aligned} \quad (122)$$

retrieves (121) by an expansion to the quartic order. Here, we simply use the fact that $\exp(ax + bx^2) = 1 + ax + (\frac{1}{2}a^2 + b)x^2 + \dots$. We thus obtain

$$c_4 = \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \sum_{m=1}^N G_c^4(j, k, l, m) p_j p_k p_l p_m , \quad (123)$$

with

$$G_c^4(j, k, l, m) = 24 \sum_{a=1}^N \sum_{b=1}^N \sum_{c=1}^N \sum_{d=1}^N v_{abcd} \left(1 - \frac{1}{2}(V_{ab}V_{cd} + V_{ac}V_{bd} + V_{ad}V_{bc}) \right) V_{aj}V_{bk}V_{cl}V_{dm} . \quad (124)$$

The higher order terms like $v_{abcdef}X^aX^bX^cX^dX^eX^f$ also contribute to the fourth order cumulant as seen by generalizing (121). In fact, c_4 receives contributions from all higher order terms. They are weighted by the coefficients $v_{abcdef} \dots$ which in general decrease rather fast. Diagramatic techniques can then be used to keep track of all terms at a given order in the systematic expansion [21].

4.6.3 Application to the quasi-gaussian case

The quasi-gaussian case where the distribution $P(\delta S)$ has the form (97) implies precise constraints on the structure of the cumulants of $P(\delta S)$ and thus on the correlation functions between the assets. Indeed, from the expression (72) giving $P(\delta S)$ up to the first correction in terms of the kurtosis, we see that $P(\delta S)$ is uniquely a function of $\frac{\delta S^2}{P^TVP}$ (where P^TVP is denoted V in (72)) only if the cumulant of order 4 is proportional to V^2 with a coefficient of proportionality which is a pure number. As a consequence, the kurtosis must be a number independent of the asset weights in the

portfolio. For this to be true, the cumulant c_4 given by (123) must factorize and is proportional to the square of the cumulant c_2 :

$$c_2 \equiv \sum_{j=1}^N \sum_{k=1}^N G_c^2(j, k) p_j p_k , \quad (125)$$

where

$$G_c^2(j, k) = \sum_{a=1}^N \sum_{b=1}^N V_{aj} V_{bk} . \quad (126)$$

The identification term by term yields

$$G_c^4(j, k, l, m) = w G_c^2(j, k) G_c^2(l, m) , \quad (127)$$

where w is arbitrary. This expression (127) together with (124) and (126) determines the particular structure of the four-asset correlations v_{abcd} in the quasi-gaussian case (97).

5 Conclusion

We have tried to demonstrate the analogies between the quantification of risks in finance and insurance and the optimization of portfolios on one hand and statistical physics concepts and methods on the other hands. The main message similar to that given long ago for random physical systems [22] is that a suitable risk assessment requires the study of the full distributions of price variations in contrast to the more standard variance approach. We have also shown how tools developed in statistical physics to address large fluctuations can be used in the optimization of portfolios. We thus hope to foster the interest of the physical community in these fascinating problems.

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FIGURE CAPTIONS

Figure 1: Return r and variance v as a function of f for $p = 1/2$ and $\lambda = 2.1$.

Figure 2: Return r as a function of v for the same values $p = 1/2$ and $\lambda = 2.1$ as in figure 1.

Figure 3a: The characteristic time $t^*(f)$ defined by (5) as a function of f for $r \geq 0$.

Figure 3b: The Sharpe parameter $Sharpe_1 = \sqrt{2}[t^*(f)]^{-1/2} = \frac{r}{\sqrt{v}}$ defined for a unit time step as a function of f .

Figure 4: Dependence of the average wealth $\langle S(1) \rangle$ and of the typical wealth $S_{pp}(1)$ as a function of f after one time step, for the case $p = 1/2$ and $\lambda = 2.1$.

Figure 5a: The Cramér function $s(x)$ given by (29) and its parabolic approximation as a function of x for $p = 0.5$.

Figure 5b: The Cramér function $s(x)$ given by (29) and its parabolic approximation as a function of x for $p = 0.95$.

Figure 6: The Cramér function $s(y \equiv \alpha\delta S) = \log(y) + 1 - y$ and its parabolic approximation $s_g(y) = -\frac{1}{2}(y - 1)^2$.